THE CONNECTION BETWEEN NORMALIZABLE AND SPECTRAL OPERATORS

BY

L. TZAFRIRI*

ABSTRACT

A necessary and sufficient condition for an operator to be normalizable is given in terms of Dunford's spectral theory.

In [7], Zaanen defined the notions of symmetrizable and normalizable operators with respect to a positive, self-adjoint fixed operator H. His definitions become evident and more transparent by taking into account that an operator is normalizable (symmetrizable) whenever there exists a certain naturally corresponding normal (self-adjoint) operator on a certain factor space of our Hilbert space (non-necessary complete) (see [7] §12, p. 226).

Using the additional assumption that H has a closed range, we prove here that an operator is normalizable (symmetrizable) with respect to H if and only if it is normal (self-adjoint) on a subspace of our basic Hilbert space provided with a new and equivalent norm.

Relying on the above-mentioned result and using an idea of Mackey [5] and Wermer [6] we carry on by proving that an operator is normalizable if and only if its product with the orthogonal projection on the afore-mentioned subspace is a spectral operator of scalar type in Dunford's sense [1]. Later we shall draw the conclusion that the product between a positive self-adjoint invertible operator and a self-adjoint operator is a spectral operator of scalar type having a real spectrum. It should be mentioned that this result is proved with no assumption of commutativity of the operators.

1. **Notation.** Our notation is essentially that of Zaanen [7], [8]. Throughout the paper $\mathfrak{X}$ will denote a fixed Hilbert space; $H \neq 0$ a bounded positive self-adjoint operator defined in $\mathfrak{X}$; $\mathscr{L}$ the null space of H; $\mathscr{M} = \mathscr{L}^{\perp}$ and P the orthogonal projection on $\mathscr{M}$. Every operator will be assumed to be bounded.

For convenience, we shall summarize here some definitions and results from [7], [8].

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First, we shall mention that

$$H = HP = PH \tag{1.1}$$

DEFINITION 1. Given an operator T, any operator $\tilde{T}$ satisfying the relation

$$(HTx, y) = (Hx, \tilde{T}y) \qquad x, y \in \mathfrak{X} \tag{1.2}$$

will be called an H-adjoint of T.

Generally, the H-adjoint of an operator is not uniquely determined. In the special case that $H = I$, the H-adjoint is uniquely determined and equal to the ordinary adjoint T^*.

DEFINITION 2. An operator T will be called H-symmetrizable if $T = \tilde{T}$, i.e.,

$$(HTx, y) = (Hx, Ty) \qquad x, y \in \mathfrak{X} \tag{1.3}$$

Obviously, T is H-symmetrizable whenever HT is self-adjoint.

DEFINITION 3. An operator T will be called H-normalizable if:

$$H\tilde{T}T = HT\tilde{T} \tag{1.4}$$

The concept of spectral operator as used here is that which was developed by Dunford in [1], [3].

2. **Spectral properties.** Let us begin with a necessary and sufficient condition for the existence of the H-adjoint.

LEMMA 4. *Assume H a positive operator with a closed range and let $H_0 = H/\mathscr{M}$ be the restriction of H to the subspace $\mathscr{M}$; then H_0 is invertible.*

Proof. Using (1.1) it follows $H\mathfrak{X} = H\mathscr{M} \subseteq \mathscr{M}$ and therefore H_0 is one-to-one, positive, self-adjoint operator. Hence, if $0 \in \sigma(H_0)$ then 0 is in the continuous spectrum of H_0, i.e. $H_0\mathscr{M} = H\mathscr{M} = H\mathfrak{X}$ is dense in $\mathscr{M}$. But, since H_0 has a closed range it is invertible.

THEOREM 5. *Let H be a positive, self-adjoint operator with a closed range. An operator T will have an H-adjoint if and only if*

$$PT = PTP \tag{2.1}$$

or, equivalently, $\mathscr{L}$ and $\mathscr{M}$ are invariant under PT.

Proof. If T has an H-adjoint $\tilde{T}$, then by (1.1)

$$(PTx, Hy) = (HPTx, y) = (HTx, y) = (Hx, \tilde{T}y) = 0$$

for every $x \in \mathscr{L}$, $y \in \mathfrak{X}$, i.e. $PT\mathscr{L} \perp H\mathfrak{X}$ thus $PT\mathscr{L} \perp \mathscr{M}$, hence $PT\mathscr{L} \subseteq \mathscr{L}$. On the other hand it is clear that $PT\mathscr{M} \subseteq \mathscr{M}$, i.e. $\mathscr{L}$ and $\mathscr{M}$ are subspaces invariant under PT and, therefore, $PT = P(PT) = (PT)P = PTP$. It should be mentioned that in this part of the proof, the fact that H has a closed range is not used.

Conversely, if (2.1) holds then

$$(HTx,y) = (HPTx,y) = (HPTPx,y) = (x,PT^*PHy) = (x,H_0H_0^{-1}PT^*Hy)$$
$$= (x,HH_0^{-1}PT^*Hy) = (Hx,H_0^{-1}PT^*Hy) \qquad x,y \in \mathfrak{X}$$

and, therefore, we will be able to put $\tilde{T} = H_0^{-1}PT^*H$.

In [7], Zaanen elucidates the definitions 2 and 3 through the factor space $\mathfrak{X}/\mathscr{L}$. Given $[x],[y] \in \mathfrak{X}/\mathscr{L}$ it can be defined a new inner product on $\mathfrak{X}/\mathscr{L}$ by putting

$$\langle [x],[y] \rangle = (Hx,y)$$

but with this new norm $\mathfrak{X}/\mathscr{L}$ does not have to be complete. If T has an H-adjoint it is easy to see that $Hx = 0$ implies $HTx = 0$ and hence we can define without ambiguity the operator $[T]$ acting in $\mathfrak{X}/\mathscr{L}$ by putting:

$$[T][x] = [Tx] \qquad [x] \in \mathfrak{X}/\mathscr{L}$$

Zaanen proved that a compact operator T is H-normalizable (symmetrizable) if and only if $[T]$ is a bounded normal (self-adjoint) operator on $\mathfrak{X}/\mathscr{L}$, provided with a new norm.

Relying on the afore-mentioned consideration and the additional assumption that H has a closed range we give here this parallel construction; by Lemma 4 $H\mathfrak{X} = H\mathscr{M} = \mathscr{M}$ i.e. $\mathscr{M}$ is an invariant subspace of H. Define a new inner product on $\mathscr{M}$ as follows:

$$\langle x,y \rangle = (Hx,y) = (H^{1/2}x, H^{1/2}y) \qquad x,y \in \mathscr{M}$$

Denoting the initial norm by $\|\cdot\|$ and the new norm on $\mathscr{M}$ by $|||\cdot|||$, we shall have

$$|||x||| = \|H^{1/2}x\| \leqq \|H^{1/2}\| \cdot \|x\| \qquad x \in \mathfrak{X}$$

but, according to the Lemma 4 $0 \notin \sigma(H_0)$ and hence $0 \notin \sigma(H_0^{1/2})$ i.e.

$$\|x\| = \|H_0^{-1/2}H_0^{1/2}x\| \leqq \|H_0^{-1/2}\| \cdot |||x||| \qquad x \in \mathfrak{X}$$

and it will follow that these two norms on $\mathscr{M}$ are equivalent and $\mathscr{M}$ with the new norm is a complete space.

By using Theorem 5 we have:

$$\langle PTx,y \rangle = (HPTx,y) = (HTx,y) = (Hx,\tilde{T}y) = (x,HP\tilde{T}y)$$
$$= (Hx,P\tilde{T}y) = \langle x,P\tilde{T}y \rangle \qquad x,y \in \mathscr{M}$$

for every operator T satisfying (2.1). Hence

$$(PT)^{+}x = P\tilde{T}x \qquad x \in \mathcal{M} \tag{2.2}$$

where $(PT)^{+}$ is the adjoint of PT within the space $\mathcal{M}$, with the new norm.

THEOREM 6. *An operator T is H-normalizable if and only if it satisfies* (2.1) *and PT is a normal operator on the Hilbert space* $\{\mathcal{M}, |||\cdot|||\}$.

Proof. If T is H-normalizable, then it has an H-adjoint and according to the Theorem 5, (2.1) is satisfied. In addition

$$\begin{aligned}(H\tilde{T}Tx,y) &= (H\tilde{T}Tx,Py) = (HTx,TPy) = (HTx,PTPy)\\ &= (Hx,\tilde{T}PTPy) = (HPx,(P\tilde{T})(PT)Py) = \langle Px,(P\tilde{T})(PT)Py\rangle \quad x,y \in \mathfrak{X}\end{aligned}$$

and similarly

$$(HT\tilde{T}x,y) = \langle Px,(PT)(P\tilde{T})Py\rangle \qquad x,y \in \mathfrak{X}$$

therefore $H\tilde{T}T = HT\tilde{T}$ whenever $(P\tilde{T})(PT)P = (PT)(P\tilde{T})P$ or, using (2.2), if and only if PT will be a normal operator on $\{\mathcal{M}, |||\cdot|||\}$.

For H-symmetrizable operators we have a similar theorem.

THEOREM 7. *An operator T is H-symmetrizable if and only if it satisfies* (2.1) *and PT is a self-adjoint operator on the space* $\{\mathcal{M}, |||\cdot|||\}$.

Proof. We have

$$(HTx,y) = (HPTPx,y) = \langle PTPx,Py\rangle \qquad x,y \in \mathfrak{X}$$

and similarly

$$(Hx,Ty) = (H\tilde{T}x,y) = \langle P\tilde{T}Px,Py\rangle \qquad x,y \in \mathfrak{X}$$

Hence $PTP = P\tilde{T}P$ if and only if T is H-symmetrizable and the proof may be finished by (2.2).

The two next theorems elucidate the connection between normalizable or symmetrizable operators and spectral operators.

THEOREM 8. *If T is an H-normalizable operator, then PT is a spectral operator of scalar type. Moreover, if T is H-symmetrizable then the spectrum of PT is real.*

Proof. According to the Theorem 6, PT is a normal operator on the space $\{\mathcal{M}, |||\cdot|||\}$ in which the new norm $|||\cdot|||$ is equivalent to the initial one $\|\cdot\|$ and, therefore, $PT/\mathcal{M}$ is a spectral operator of scalar type on $\{\mathcal{M}, \|\cdot\|\}$. If T is H-symmetrizable then by the theorem 7, PT will be self-adjoint on $\{\mathcal{M}, |||\cdot|||\}$ and, therefore, scalar with real spectrum on $\{\mathcal{M}, \|\cdot\|\}$.

According to Theorem 5 $PT\mathcal{L} = PTP\mathcal{L} = \{0\}$ and we can conclude, using the Dunford and Schwartz [3] Theorem XVI–5–3, that PT is a spectral operator of scalar type on the whole space $\mathfrak{X}$. If T is H-symmetrizable, naturally, $\sigma(PT)$ will be a real set.

In order to prove the converse of the previous theorem we shall use a result of Mackey [5].

THEOREM 9. *Let P be a self-adjoint projection in $\mathfrak{X}$ and T an operator such that PT is spectral of scalar type and (2.1) is satisfied. Then, there exists a positive, self-adjoint operator H, having a closed range and such that T is an H-normalizable operator satisfying $H\mathfrak{X} = P\mathfrak{X}$ and*

$$H = PH = HP. \tag{2.3}$$

Moreover, if $\sigma(PT)$ is a real set, then T is H-symmetrizable.

Proof. First, let us remark that the restriction of PT to $\mathscr{M} = P\mathfrak{X}$ is a spectral operator of scalar type on $\mathscr{M}$ and $\sigma(PT/\mathscr{M})$ is real whenever $\sigma(PT)$ is real too.

By Mackey [5], Theorem 55 (see also Wermer [6]) we can define a new norm $|||\cdot|||$ on $\mathscr{M}$ (associated with a new inner product $\langle\cdot,\cdot\rangle$) which is equivalent to the initial one and such that $PT/\mathscr{M}$ is a normal operator on $\{\mathscr{M}, |||\cdot|||\}$. If $\sigma(PT/\mathscr{M})$ is real then $PT/\mathscr{M}$ is self-adjoint on $\{\mathscr{M}, |||\cdot|||\}$.

Using [2], Lemma X. 2.2, we can suppose that the new inner product is given by

$$\langle x, y\rangle = (Bx, y) \qquad x, y \in \mathscr{M}$$

where B is a positive, self-adjoint operator on $\{\mathscr{M}, \|\cdot\|\}$, having a bounded everyewhere defined inverse. Hence, $H = BP$ is a positive, self-adjoint operator on $\mathfrak{X}$ satisfying (2.3) and $H\mathfrak{X} = P\mathfrak{X} = \mathscr{M}$ since

$$(Hx, x) = (BPx, x) = (PBPx, x) = (BPx, Px) = |||Px|||^2 \geqq 0 \qquad x \in \mathfrak{X}$$

Now, let $\{x_n\}\, x_n \in \mathfrak{X};\ n = 1, 2, \cdots$ be such that

$$\lim_{n\to\infty} Hx_n = y \tag{2.4}$$

Then $Py = y$ i.e. $y \in \mathscr{M}$. Applying B^{-1}on (2.4) we shall get

$$\lim_{n\to\infty} Px_n = B^{-1}y$$

and therefore

$$\lim_{n\to\infty} Hx_n = HB^{-1}y$$

and, consequently $y = HB^{-1}y$, i.e. H has a closed range.

Taking into account (2.3) we can remark that P is just the orthogonal projection on the orthogonal complement of the null space of H; hence by Theorems 6 and 7 T is H-normalizable and if $\sigma(PT)$ is real then it is H-symmetrizable.

COROLLARY 10. *Let A and B be self-adjoint operators; if A is positive and has an inverse (defined everywhere), then the product AB is a spectral operator of scalar type, with real spectrum.*

Proof. Denote $S = AB$; it follows that $B = A^{-1}S$ is self-adjoint and from the observation made after the definition 2 it follows that S is A^{-1}-symmetrizable (A^{-1} is positive, self-adjoint and invertible too). Therefore, the orthogonal projection on the orthogonal complement of the null space of A^{-1} coincides with the identity I. By the theorem 8 S is a spectral operator of scalar type and $\sigma(S)$ is real.

Every normal operator N can be decomposed as follows:

$$N = N_1 + iN_2$$

where $N_1 = (N + N^*)/2$ and $N_2 = (N - N^*)/2i$ are commutative self-adjoint operators. A similar decomposition exists for H-normalizable operators too.

THEOREM 11. *Let T be H-normalizable. Then, there exist two H-symmetrizable operators T_1 and T_2 satisfying:*

(a) $T = T_1 + iT_2$

(b) $PT_1T_2 = PT_2T_1$

(c) *If S_1, S_2 are H-symmetrizable operators such that $T = S_1 + iS_2$ and $PS_1S_2 = PS_2S_1$ then $PS_i = PT_i$, $i = 1,2$.*

Proof. We can put

$$T_1 = \frac{T + \tilde{T}}{2}; \quad T_2 = \frac{T - \tilde{T}}{2i}$$

Obviously, T_1 and T_2 are H-symmetrizable and $HT_1T_2 = HT_2T_1$. Using (1.1) we get $H(PT_1T_2 - PT_2T_1) = 0$ i.e. $(PT_1T_2 - PT_2T_1) \in \mathscr{L} \cap \mathscr{M} = \{0\}$ and, hence (b) is satisfied. If (c) holds for some S_1 and S_2 then by the Theorem 8 the operators PT_1, PT_2, PS_1 and PS_2 will be spectral of scalar type and their spectrum will be real. But

$$PT = PT_1 + iPT_2 = PS\ + iPS_2$$

and using the Theorem 5

$$(PT_1)(PT_2) = (PT_1P)T_2 = PT_1T_2 = PT_2T_1 = (PT_2)(PT_1)$$

and similarly $(PS_1)(PS_2) = (PS_2)(PS_1)$. According to Foguel [4] (Theorem 1. p. 59) we get $PS_i = PT_i$, $i = 1,2$.

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